

Exploitation of the Dissipation Inequality in General Relativistic Continuum Thermodynamics *

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Abstract

The balance equations of energy-momentum and spin together with Einstein's field equations are investigated by the Liu procedure to find constraints for the constitutive equations in such a way that the Second Law is satisfied. Special cases such as spinless systems and curvature insensitive materials are shortly discussed.

1 Introduction

The balance equations of continuum physics are formulated for arbitrary materials, that means, they present an underdetermined system of differential equations which need for its solution additional equations, the material equations which made the system of the balance equations solvable for a special material. There are two different procedures: The material equations are presupposed¹ and solved together with the balances asking later, if the Second Law is satisfied for all positions and times. The second method takes the Second Law into consideration before solving the balances² asking for material equations supplementing the balances equations so that their common solution is in agreement with the Second Law.

Here, we deal with the second method which is well known in non-relativistic material theory [1, 2], but is up to now not strictly developed in general-relativistic constitutive theory, except of an attempt [3]. A reason for that delay may be caused by the question, how to handle Einstein's field equations in this framework. Precondition for taking Einstein's field equations into consideration is that those and the balance equations are formulated in a compatible form³. Having done this, the

*Dedicated to Gérard A. Maugin on the occasion of his 70th birthday

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¹The practitioner knows his material and chooses the correct ansatz.

²The theorist looks for a class of materials in agreement with the Second Law.

³The energy-momentum tensors are different the in field and balance equations, that needs a modified Belinfante-Rosenfeld procedure, here also taking the field equations into consideration [4].

question arises, if Einstein's field equations fit in the well known procedure of generating the material class being compatible with the Second Law⁴. This is the case due to the fact that Einstein's gravitational equations are differential equations of second order for the metric. Thus, our approach is completely in accordance with Maugin's opinion [6]⁵. Encouraged by his remarks, we derive the Liu equations and the dissipation inequality for the combined system of energy-momentum and spin balances –that are the Mathisson-Papapetrou equations– and Einstein's field equations. Special cases are considered: spinless systems and curvature insensitive materials.

The paper is organized as follows: After some introductory considerations on tetrads and connexions, the balances and the gravitational equations are written down in (3+1)-decomposition. State spaces of first and second order are introduced. The Liu equations and the dissipation inequality are derived for a first order state space material. A discussion finishes the paper.

2 Tetrads and Connexions

Measuring values produced by measuring devices need for their description a frame of reference which is locally spanned by tetrads $\{e_i^A\}$, $A, i = 1, \dots, 4$, consisting of three space-like vectors and one time-like vector numbered by A ⁶. These vectors represent a standard clock and three standard measuring rods⁷. The field of tetrads $\{e_i^A(x^a)\}$ is called an *observer field* or a *system of reference*.

The corresponding contravariant components of the tetrad e_i^A , denoted by e_A^k , are defined by

$$e_A^k e_i^A = \delta_i^k, \quad \longleftrightarrow \quad e_A^k e_k^B = \delta_A^B. \quad (1)$$

The tetrad e_A^k is presupposed to be *ortho-normalized* with respect to the metric g_{ik} of a pseudo-Riemannian space

$$g_{ik}(x^a) e_A^i(x^a) e_B^k(x^a) = \eta_{AB}, \quad (\eta_{AB}) := \begin{pmatrix} -1 & \mathbf{0} \\ \mathbf{0}^\top & 1 \end{pmatrix}. \quad (2)$$

The matrix (η_{AB}) has the shape of the Minkowski metric. Multiplying (2)₁ by $e_i^A e_k^B$ and using (1) results in

$$g_{ik} = \eta_{AB} e_i^A e_k^B \quad \longleftrightarrow \quad g^{mn} = \eta^{CD} e_C^m e_D^n. \quad (3)$$

The 16 components $\{e_A^i\}$ or $\{e_i^A\}$, are restricted by 10 constraints, (2)₁ or (3). Thus for a given metric, 10 components of the system of reference are fixed.

⁴for the non-relativistic case see [5]

⁵Maugin: "First, from the formal point of view, we may consider Einstein's theory of general relativity as a generalized continuum theory of the second gradient of the space-time metric.... Second, the Maxwell stress and other such electromagnetic stress tensors....are none other than purely spatial parts of the space-time energy-momentum tensor....Finally, while we can have some doubt with special relativity, it is clear that general relativity is a continuum theory from the start. This is more than enough to ponder the formulation of continuum thermodynamics in its framework."

⁶The tetrad representation is well known. For more details we refer to [7, 8].

⁷That is the reason why we introduce tetrads.

The tetrads depend on the event, whereas (η_{AB}) does not.

The tetrad components e_A^i and e_i^A , respectively, are matrices transforming tensor components at each point of the pseudo-Riemannian space. For a tensor of first order, we have the following

$$a_B := e_B^j a_j \quad \longleftrightarrow \quad a_j = e_j^B a_B, \quad (4)$$

and generally, we obtain for tensors of higher order

$$T^{A\dots B\dots} = T^{c\dots d\dots} e_c^A e_d^B, \quad T^{a\dots b\dots} = T^{C\dots D\dots} e_C^a e_D^b \dots \quad (5)$$

We now consider local transformations $L_A^B(x^a)$ of the tetrad components

$$\text{at } x^a : \quad \overset{*}{e}_j^B = L_A^B(*\diamond) \overset{\circ}{e}_j^A, \quad L_B^C(\diamond*) L_A^B(*\diamond) = \delta_A^C. \quad (6)$$

Inserting (6) into (4), results in

$$\overset{\circ}{a}_A = L_A^B(*\diamond) \overset{*}{a}_B. \quad (7)$$

The properties of the matrix L_A^B are connected to the ortho-normalized tetrads: Inserting (6) into (3) results in

$$e_i^A \equiv \overset{*}{e}_i^A, \quad g_{ik} = \eta_{AB} L_C^A \overset{\circ}{e}_i^C L_D^B \overset{\circ}{e}_k^D. \quad (8)$$

The Principle of Relativity now asserts that all systems of reference which are compatible with the given metric g_{ik} are equivalent to each other, i.e. that $\overset{\circ}{e}_j^A$ in (6) belongs also to an ortho-normalized tetrad as e_j^B in (3) does. According to (8), this demand is achieved by the setting

$$\eta_{AB} L_C^A(x^a) L_D^B(x^a) \doteq \eta_{CD}. \quad (9)$$

This equation is the definition for $L_B^A(x^a)$ to be a Lorentz transformation at each point (x^a) . Thus, $L_B^A(x^a)$ belongs to the group of local space-time dependent Lorentz transformations and transforms an orthogonal-normalized tetrad into an orthogonal-normalized one at each x^a .

In this paper, we consider the (anholonomic) tetrad representation of General Relativity, instead of the (holonomic) metric one. Accordingly, one has to work in a framework, in which the *Principle of General Relativity* is realized by the covariance of the basic laws with respect to local Lorentz transformations, instead of general coordinate transformations (what requires the definition of the Lorentz connexion)⁸. This is the relativistic version of the *Principle of*

⁸To avoid misunderstandings, it should be mentioned that the tetrads here introduced as anholonomic coordinates differ as well from those introduced in tetrad theories of gravitation, where they are considered as gravitational potentials, as from those ones introduced as so-called directors for describing the spin of the relativistic continua by Maugin and Eringen [9].

Material-Frame Indifference (or objectivity)⁹.

The covariant derivative $a_B^A|_m$ is related to the partial one $a_{B,m}^A$ by the connexion Ω_{mP}^A as follows¹⁰

$$a_B^A|_m = a_{B,m}^A + \Omega_{mP}^A a_B^P - \Omega_{mB}^Q a_Q^A, \quad (10)$$

and the connexion Ω_{mB}^Q is defined by

$$0 = e_i^A|_m - \Gamma_{mi}^q e_q^A + \Omega_{mQ}^A e_i^Q. \quad (11)$$

Here Γ_{mk}^q is the connexion belonging to the metric g_{ik} in (3)₁ [16]

$$\Gamma_{mi}^b = \frac{1}{2}(g_{mq,i} + g_{iq,m} - g_{mi,q})g^{bq}. \quad (12)$$

According to (3)₁, (11) and (12), the connexion is a function of the tetrads and their first partial derivatives

$$\Omega_{mQ}^A = \omega_{mQ}^A(e_j^D, e_{j,k}^D). \quad (13)$$

3 Balance Equations

In this section, we start out with the well-known general-relativistic balance and field equations of continuum physics. These balance equations are generated from the special-relativistic balances by applying Einstein's *Principle of Equivalence*, mathematically realized by the Principle of Minimal Coupling which asserts that the partial derivatives in the special-relativistic balances have to be replaced by covariant ones with respect to local Lorentz transformations [17].

3.1 The general case

Thus, we obtain in a pseudo-Riemannian space the following thermodynamical balances:

Particle number density:

$$N^A|_A = 0, \quad N^A := \frac{1}{c^2} n u^A. \quad (14)$$

Here, u^A is the material 4-velocity of the particles and c the vacuum velocity of light.

Energy-momentum:

$$T^{AB}|_B = f^A. \quad (15)$$

⁹For the special-relativistic theory, being covariant under rigid Lorentz transformations, this was already shown by Maugin and Eringen [10]. Later on in a series of papers in JMP [11], Maugin considered the space-time covariant description of constitutive equations also in general relativity as the relativistic replica of the classical notion of objectivity where covariance means: covariance with respect to general coordinate or local Lorentz transformations. For the non-relativistic case see [12, 13, 14, 15].

¹⁰for more details concerning the tetrad calculus see here and in the sequel [8]

Here, T^{AB} is the energy-momentum tensor and f^A the external 4-force density.

Spin:

$$S_{BC}^A|_A = m_{BC} = -m_{CB}. \quad (16)$$

Here, S_{BC}^A is the spin tensor and m_{BC} the external 4-momentum density.

Entropy:

$$S^A|_A - \varphi = \sigma \geq 0 \quad (17)$$

with the 4-entropy vector S^A , the entropy supply φ and the non-negative entropy production σ ¹¹. This inequality represents the Second Law of classical field theories, called the *dissipation inequality* [18, 19].

Here, the constitutive theory is especially developed in the framework of GRT¹². Consequently, we have to take into consideration Einstein's

Field Equations:[16]

$$R_{AB} - \frac{1}{2}\eta_{AB}R = \kappa\Theta_{AB}. \quad (18)$$

Here, R_{AB} is the symmetric Ricci tensor and R the curvature scalar

$$R_{AB} := R_{ACB}^C, \quad R := R_A^A, \quad (19)$$

and R_{ADB}^C is the curvature tensor of a pseudo-Riemannian space

$$\begin{aligned} R_{ADB}^C &= R_{mni}^b e_b^C e_A^m e_D^n e_B^i = \\ &= (\Gamma_{mi,n}^b - \Gamma_{mn,i}^b + \Gamma_{mi}^j \Gamma_{nj}^b - \Gamma_{mn}^j \Gamma_{ij}^b) e_b^C e_A^m e_D^n e_B^i. \end{aligned} \quad (20)$$

As in the appendix 8 derived, the curvature tensor is linear in the second partial derivatives of the tetrads

$$R_{mni}^b = E_{mniG}^{buvw} (e_j^A) e_u^G{}_{,vw} + F_{mni}^b (e_j^A, e_{j,p}^A), \quad (21)$$

and depends also on their first derivatives and on the tetrads themselves. Completely written in tetrad components, we have

$$R_{ADB}^C = G_{ADBG}^{Cuvw} (e_j^A, e_{j,p}^A) e_u^G{}_{,vw} + H_{ADB}^C (e_j^A, e_{j,p}^A). \quad (22)$$

In accordance with (18), the *gravitation generating energy-momentum tensor* Θ_{AB} has to be symmetric and divergence-free

$$\Theta_{AB} = \Theta_{BA} \wedge \Theta^{AB}|_B = 0. \quad (23)$$

The field equations (18) yield by multiplication with η^{CA} and by use of (19)₂

$$R = -\kappa\Theta_B^B \longrightarrow R_{AB} = \kappa\Theta_{AB} - \frac{1}{2}\eta_{AB}\kappa\Theta_C^C. \quad (24)$$

¹¹All these quantities are densities.

¹²General Relativity Theory

Obvious is, that the gravitation generating energy-momentum tensor Θ_{AB} is different from that in the energy-momentum balance, because T_{AB} does not satisfy (23). Therefore the question arises: what is the relation between Θ_{AB} and T_{AB} ?¹³ An additional question is: are the balance equations (15) and (16) compatible with the field equations (18)? To answer this question, we refer to [4], where we proved the following

■ **Proposition:** If we presuppose that in GRT energy-momentum and spin satisfy balance equations –(15) and (16)– then the gravitation generating energy-momentum tensor in Einstein’s field equations is

$$\Theta^{AB} = T^{AB} - \frac{1}{2}\Sigma^{CAB}|_C, \quad (25)$$

$$\Sigma^{CAB} := S^{CAB} + S^{ABC} + S^{BAC}, \quad (26)$$

and the external force density and the momentum density are

$$f^A = -\frac{1}{2}R_{CDE}^A S^{CDE}, \quad (27)$$

$$m^{BA} = 2T^{[BA]}. \quad (28)$$

Consequently, the corresponding balance equations of energy-momentum and spin are the Mathisson-Papapetrou equations [21, 22]

$$T^{AB}|_A = -\frac{1}{2}R_{CDE}^B S^{CDE}, \quad (29)$$

$$S^{CAB}|_C = 2T^{[AB]}. \quad \blacksquare \quad (30)$$

The gravitational equations (24) of General Relativistic Continuum Theory (GRCT) become

$$\begin{aligned} R^{AB} + \frac{1}{2}\eta^{AB}\kappa\Theta_D^D &= \kappa\left(T^{AB} - \frac{1}{2}(S^{CAB} + S^{ABC} + S^{BAC})|_C\right) = \\ &= \kappa\left(T^{(AB)} - \frac{1}{2}(S^{ABC} + S^{BAC})|_C\right). \end{aligned} \quad (31)$$

The trace of the gravitation generating energy-momentum tensor is according to (31) and (24)

$$\Theta_D^D = T_D^D - S_D^{DC}|_C. \quad (32)$$

Consequently, the Ricci tensor in (31) becomes

$$R^{AB} = \kappa\left(T^{(AB)} - \frac{1}{2}(S^{ABC} + S^{BAC})|_C - \frac{1}{2}\eta^{AB}\left[T_D^D - S_D^{DC}|_C\right]\right) \quad (33)$$

which is determined by the symmetric part of the energy-momentum tensor and derivatives of the spin tensor. The field equations (31) become

$$(S^{ABC} + S^{BAC} + \eta^{AB}S_D^{DC})|_C + \frac{2}{\kappa}R^{AB} = 2T^{(AB)} - \eta^{AB}T_D^D. \quad (34)$$

¹³For the Special Relativity Theory, this question is answered in [20].

According to (34), all derivatives appearing in the field equations are on the lhs. This form of the field equations is essential for the exploitation of the dissipation inequality (17) by the Liu procedure which is discussed in sect.5.

3.2 (3+1)-Decomposition

For the exploitation of the dissipation inequality, we need (29), (30) and (34) in (3+1)-decomposition. Starting out with the particle flux, the energy-momentum tensor, the spin tensor and the 4-entropy [23, 24, 25], we obtain

$$N^A = \frac{1}{c^2} n u^A, \quad (35)$$

$$T^{AB} = \left(\frac{1}{c^4} e u^B + \frac{1}{c^2} p^B \right) u^A + \frac{1}{c^2} u^B q^A + t^{AB}, \quad (36)$$

$$S^{CAB} = \left(\frac{1}{c^2} s^{AB} + \frac{1}{c^4} u^{[A} \Xi^{B]} \right) u^C + \frac{1}{c^2} u^{[A} \Xi^{B]C} + s^{CAB}, \quad (37)$$

$$S^A = \frac{1}{c^2} s u^A + s^A. \quad (38)$$

In the sequel, we split the fields of the (3+1)-decomposition into two classes. Using the projector h^{AB} onto the sub-space perpendicular to the 4-velocity u^A

$$h^{AB} := \eta^{AB} - \frac{1}{c^2} u^A u^B = h^{BA}, \quad (39)$$

the first class is defined by

$$n := N^A u_A, \quad (40)$$

$$e := T_{AB} u^A u^B, \quad \varepsilon := e/c^2, \quad (41)$$

$$s_{AB} := S_{EF}^C h_A^E h_B^F u_C, \quad (42)$$

$$\Xi_A := 2S_{EF}^C u_C u^E h_A^F. \quad (43)$$

Here n is the *particle density*, e the *energy density*, s_{AB} the *spin density* and finally Ξ_A the *spin density vector*.

The second class of fields of the (3+1)-decomposition is defined by [23, 25]

$$t^{AB} := h^{AE} T_{EF} h^{FB}, \quad (44)$$

$$p^A := h^{AE} T_{FE} u^F, \quad (45)$$

$$q^A := h^{AE} T_{EF} u^F, \quad (46)$$

$$s_{AB}^C := S_{EF}^G h_A^E h_B^F h_G^C, \quad (47)$$

$$\Xi_B^A := 2S_{EF}^G h_G^A u^E h_B^F, \quad (48)$$

$$s := S^A u_a, \quad s^A := S^B h_B^A. \quad (49)$$

Here t^{AB} is the *stress tensor*, p^A the *momentum flux density*, q^A the *energy flux density*, s_{AB}^C the *couple stress*, Ξ_B^A the *spin stress*, s the *entropy density* and s^A the *entropy flux*.

We now write down the balance equations in (3+1)-decomposition. According to (29) and (36), we obtain the energy-momentum balance

$$\begin{aligned} \frac{1}{c^2} \left(p|_A^B u^A + u^B q|_A^A \right) + t|_A^{AB} + \frac{1}{c^2} \left(p^B u|_A^A + u|_A^B q^A \right) + \frac{1}{c^4} (e u^B u^A)|_A = \\ = -\frac{1}{2} R_{CDE}^B S^{CDE}. \end{aligned} \quad (50)$$

The spin balance results from (30) and (37)

$$\frac{1}{c^2} u^{[A} \Xi^{B]C} + s|_C^{CAB} + \frac{1}{c^2} u|_C^{[A} \Xi^{B]C} + \left[\left(\frac{1}{c^2} s^{AB} + \frac{1}{c^4} u^{[A} \Xi^{B]} \right) u^C \right]|_C = 2T^{[AB]}, \quad (51)$$

and from (17) and (38) follows the dissipation inequality

$$\frac{1}{c^2} s|_A u^A + s|_A^A + \frac{1}{c^2} s u|_A^A \geq \varphi. \quad (52)$$

The field equations (34) by use of (37) result in a long-winded expression

$$\begin{aligned} \frac{1}{c^2} u^{[B} \Xi^{C]A} + s|_C^{ABC} + \frac{1}{c^2} u|_C^{[B} \Xi^{C]A} + \left[\left(\frac{1}{c^2} s^{BC} + \frac{1}{c^4} u^{[B} \Xi^{C]} \right) u^A \right]|_C + \\ + \frac{1}{c^2} u^{[A} \Xi^{C]B} + s|_C^{BAC} + \frac{1}{c^2} u|_C^{[A} \Xi^{C]B} + \left[\left(\frac{1}{c^2} s^{AC} + \frac{1}{c^4} u^{[A} \Xi^{C]} \right) u^B \right]|_C + \\ + \eta^{AB} \left\{ \frac{1}{c^2} u_{[D} \Xi^{C]D} + s_D^{DC} |_C + \frac{1}{c^2} u_{[D} |_C \Xi^{C]D} + \left[\left(\frac{1}{c^2} s_D^C + \frac{1}{c^4} u_{[D} \Xi^{C]} \right) u^D \right]|_C \right\} + \\ + \frac{2}{\kappa} R^{AB} = 2T^{(AB)} - \eta^{AB} T_D^D. \end{aligned} \quad (53)$$

The system of differential equations (50) to (53) is valid for arbitrary materials. For solving it, we need constitutive equations and the Second Law (52) has to be taken into account.

4 State Spaces

4.1 Basic and constitutive fields

The balances (14), (15) and (16) and the field equations (18) represent equations to calculate the 58 fields $N^A, T^{AB}, S_{BC}^A, S^A$ and e_i^A by taking the constraint (3) into consideration¹⁴. Additionally, we have a further constraint by the dissipation inequality (17). Balances and field equations are valid for arbitrary materials, e.g. non-Newtonian fluids, spin-fluids, solids, liquid crystals. To solve the system of differential equations combined of the balances and the field equations, we have to introduce constitutive equations. Therefore, we

¹⁴4+16+24+4+10=58. The 10 g_{ik} are replaced by 10 bilinear combinations of the 16 e_i^A according to (3).

have to split these 58 fields into those which represent the independent variables, called the *basic fields*¹⁵, and those which are material dependent ones, named the *constitutive fields* [26].

Like in non-relativistic continuum physics, we choose as basic fields the first class (40) to (43) and additional the field of tetrads which according to (6)₁ are tensors of first order in the tetrad indices

$$\mathbf{z} = (n, u^A, e, s_{AB}, \Xi_A, e_i^A) =: (\mathbf{w}, e_i^A). \quad (54)$$

These basic variables do not contain any derivative. Taking the normalization of the 4-velocity

$$u^A u_A = c^2 \quad \longrightarrow \quad u^A u_{A|B} = 0 \quad (55)$$

and (3) into account, the number of the basic fields in (54) is 21.¹⁶

According to the splitting into basic and constitutive fields, we now have to introduce 58 - 21 = 37 constitutive fields¹⁷. These are

$$\mathbf{M} = (t^{AB}, p^A, q^A, s_{AB}^C, \Xi_B^A, S^A). \quad (56)$$

Basic and constitutive fields are tensors of different order, that means, they are covariantly defined.

4.2 Constitutive mappings

For describing constitutive properties, we have to introduce *material mappings*¹⁸ which connect the constitutive fields (56), \mathbf{M} , with the basic fields (54), \mathbf{z} , thus characterizing the material. Obvious is, that the basic fields as the independent variables do not describe materials sufficiently in non-equilibrium. Consequently, we have to extend \mathbf{z} by additional variables $\boldsymbol{\zeta}$ consisting of derivatives of the basic fields¹⁹

$$\boldsymbol{\zeta} = (\mathbf{z}_{|m}, \mathbf{z}_{|m|n}, \dots). \quad (57)$$

The order of the additional derivatives which are enclosed in $\boldsymbol{\zeta}$ depends on the considered material. Consequently, the *set of constitutive properties* (56) depends on $(\mathbf{z}, \boldsymbol{\zeta})$. which is called the covariantly defined *state space*²⁰ and which spans the domain of the material mapping \mathcal{M}

$$\mathbf{M} = \mathcal{M}(\mathbf{z}, \boldsymbol{\zeta}). \quad (58)$$

¹⁵often named the *wanted fields*

¹⁶1+3+1+3+3+10=21

¹⁷9+3+3+9+9+4=37

¹⁸also called constitutive mappings [26]

¹⁹An other possibility of extending the basic variables is to take instead of derivatives constitutive fields into the extension, e.g. t^{AB} and q^A , see: Extended Thermodynamics [28, 29].

²⁰or constitutive space

Its range \mathbf{M} is spanned by the constitutive properties (56). The entire equation (58) is known as the *constitutive* or *material equation*

$$\mathbf{M} = \mathcal{M}(\mathbf{z}, \mathbf{z}_{|m}, \mathbf{z}_{|m|n}, \dots). \quad (59)$$

If we use partial derivatives instead of covariant ones, the connexion has to be added to the additional variables

$$\zeta' = (\mathbf{z}_{,m}, \mathbf{z}_{,mn}, \dots, \Omega_{mA}^B, \Omega_{mA,n}^B, \dots), \quad (60)$$

and the constitutive equation becomes [27]

$$\mathbf{M} = \mathcal{M}(\mathbf{z}, \mathbf{z}_{,m}, \mathbf{z}_{,mn}, \dots, \Omega_{mA}^B, \Omega_{mA,n}^B, \dots). \quad (61)$$

According to (13), the connexion itself depends on the tetrads and their first partial derivatives which are included in the $(\mathbf{z}_{,m}, \mathbf{z}_{,m,n}, \dots)$. Consequently, Ω_{mA}^B can be cancelled in the set of the state space variables

$$\mathbf{M} = \mathcal{M}(\mathbf{z}, \mathbf{z}_{,m}, \mathbf{z}_{,m,n}, \dots). \quad (62)$$

In this form of the constitutive equation, the state space does not consist of tensors in contrast to (59), but it contains all non-tensorial quantities so that by combination of them, the tensorial constitutive equation (59) can be rediscovered. Here, we prefer the non-tensorial representation of the constitutive equation, because curvature and Ricci tensor are functions of state space variables which according to (22) are partial derivatives.

4.3 First and second derivatives state spaces

Taking (11)₂ into account, the additional variables (57) of first derivative state spaces are

$$\zeta'^I \equiv (\mathbf{z}_{,m}) = (n_{,m}, u_{A,m}, \varepsilon_{,m}, s_{AB,m}, \Xi_{A,m}, e_i^A{}_{,m}) = (\mathbf{w}_{,m}, e_i^A{}_{,m}). \quad (63)$$

Consequently, the state space of first partial derivatives becomes

$$\begin{aligned} \mathcal{Z}^I &= (\mathbf{z}, \mathbf{z}_{,m}) = (\mathbf{w}, \mathbf{w}_{,m}, e_i^A, e_i^A{}_{,m}) = \\ &= (n, u^A, e, s_{AB}, \Xi_A, e_i^A, n_{,m}, u_{A,m}, \varepsilon_{,m}, s_{AB,m}, \Xi_{A,m}, e_i^A{}_{,m}). \end{aligned} \quad (64)$$

The state space \mathcal{Z}^I is a local object defined at x^a , spanned by independent variables –the wanted basic fields and their first derivatives– generating locally the constitutive fields (56) by the material mapping (58).

The additional variables of second order are

$$\begin{aligned} \zeta'^{II} &\equiv (\mathbf{z}_{,m}, \mathbf{z}_{,mn}) = (\mathbf{w}_{,m}, \mathbf{w}_{,mn}, e_i^A{}_{,m}, e_i^A{}_{,mn}) = \\ &= (\zeta'^I, n_{,mn}, u_{A,mn}, \varepsilon_{,mn}, s_{AB,mn}, \Xi_{A,mn}, e_i^A{}_{,mn}), \end{aligned} \quad (65)$$

resulting in the state space of second order

$$\mathcal{Z}^{II} = \left(n, u^k, e, s_{AB}, \Xi_A, e_i^A, n_{,m}, u_{A,m}, \varepsilon_{,m}, s_{AB,m}, \Xi_{A,m}, e_{i,m}^A, \right. \\ \left. n_{,mn}, u_{A,mn}, \varepsilon_{,mn}, s_{AB,mn}, \Xi_{A,mn}, e_{i,mn}^A \right). \quad (66)$$

Having introduced the concepts of state space and material mapping, we are able to consider the system of differential equations (50) to (53) on the chosen state spaces.

4.4 Balances and field equations on the state space

We especially consider a state space of first order (64) and write down the system (50) to (53) in which expressions like $\square^{CB}|_A$ appear. These expressions have according to (4) and (10) the form

$$\square^{CB}|_A = \square^{CB}|_j e_A^j = \left(\square_{,j}^{CB} + \Omega_{jQ}^C \square^{QB} + \Omega_{jQ}^B \square^{CQ} \right) e_A^j. \quad (67)$$

Taking (13) into account, we obtain

$$\square^{CB}|_A = \square_{,j}^{CB} e_A^j + \{ \square_A^{CB} \}(z, z_{|m}), \quad (68)$$

$$\{ \square_A^{CB} \}(z, z_{|m}) := \left(\Omega_{jQ}^C \square^{QB} + \Omega_{jQ}^B \square^{CQ} \right) e_A^j. \quad (69)$$

Because of (13) and the chosen state space of first order, the bracket symbol $\{ \square_A^{CB} \}$ does not depend on higher derivatives. The partial derivative has the following meaning, distinguishing between basic and additional variables according to (64)

$$\square^{CB}_{,j} = \left(\frac{\partial \square^{CB}}{\partial w^k} w^k_{,j} + \frac{\partial \square^{CB}}{\partial w^k_{,m}} w^k_{,mj} + \frac{\partial \square^{CB}}{\partial e_k^Q} e_k^Q_{,j} + \frac{\partial \square^{CB}}{\partial e_k^Q_{,m}} e_k^Q_{,mj} \right). \quad (70)$$

By use of (70), (68) and (22), (50) results in the *energy-momentum balance on the state space*

$$\begin{aligned} & \left(\frac{1}{c^2} \left(\frac{\partial p^B}{\partial w^k_{,m}} u^A + \frac{\partial q^A}{\partial w^k_{,m}} u^B \right) + \frac{\partial t^{AB}}{\partial w^k_{,m}} \right) e_A^j w^k_{,mj} + \\ & + \left(\frac{1}{c^2} \left(\frac{\partial p^B}{\partial e_k^Q_{,m}} u^A + \frac{\partial q^A}{\partial e_k^Q_{,m}} u^B \right) + \frac{\partial t^{AB}}{\partial e_k^Q_{,m}} \right) e_A^j e_k^Q_{,mj} + \\ & + \frac{1}{2} S^{CDE} G_{CDEQ}^{Bkmj} (e_q^A, e_{q,p}^A) e_k^Q_{,mj} = \\ = & - \left(\frac{1}{c^2} \left(\frac{\partial p^B}{\partial w^k} u^A + \frac{\partial q^A}{\partial w^k} u^B \right) + \frac{\partial t^{AB}}{\partial w^k} \right) e_A^j w^k_{,j} - \\ & - \left(\frac{1}{c^2} \left(\frac{\partial p^B}{\partial e_k^Q} u^A + \frac{\partial q^A}{\partial e_k^Q} u^B \right) + \frac{\partial t^{AB}}{\partial e_k^Q} \right) e_A^j e_k^Q_{,j} - \\ & - \frac{1}{c^2} \left(\{p_A^B\} u^A + u^B \{q_A^A\} \right) - \{t_A^{AB}\} - \frac{1}{2} S^{CDE} H_{CDE}^B (e_j^A, e_{j,p}^A) - \\ & - \frac{1}{c^2} \left(p^B u_{|A}^A + u_{|A}^B q^A \right) + \frac{1}{c^4} (e u^B u^A)|_A. \end{aligned} \quad (71)$$

The lhs of the energy-momentum balance (71) is linear in the higher derivatives $w^k{}_{,mj}$ and $e_k^Q{}_{,mj}$, whereas no higher derivatives appear on the rhs which depends on the state space variables of the first order state space (64).

Starting out with (52), the dissipation inequality becomes with (68)

$$\frac{1}{c^2} s_{,j} e_A^j u^A + s^A{}_{,j} e_A^j + \{s_A^A\} \geq \varphi - \frac{1}{c^2} s u_A^A. \quad (72)$$

By taking (70) into account, we obtain the *dissipation inequality on the state space*

$$\begin{aligned} & \left(\frac{1}{c^2} \frac{\partial s}{\partial w^k{}_{,m}} u^A + \frac{\partial s^A}{\partial w^k{}_{,m}} \right) e_A^j w^k{}_{,mj} + \left(\frac{1}{c^2} \frac{\partial s}{\partial e_k^Q{}_{,m}} u^A + \frac{\partial s^A}{\partial e_k^Q{}_{,m}} \right) e_A^j e_k^Q{}_{,mj} \geq \\ & \geq - \left(\frac{1}{c^2} \frac{\partial s}{\partial w^k} u^A + \frac{\partial s^A}{\partial w^k} \right) e_A^j w^k{}_{,j} - \left(\frac{1}{c^2} \frac{\partial s}{\partial e_k^Q} u^A + \frac{\partial s^A}{\partial e_k^Q} \right) e_A^j e_k^Q{}_{,j} + \\ & + \varphi - \{s_A^A\} - \frac{1}{c^2} s u_A^A. \end{aligned} \quad (73)$$

The pretty long-winded energy-momentum balance (71) and the dissipation inequality (73) can be written in a symbolic form

$$\text{energy-momentum:} \quad {}^{11}\mathcal{A}_k^{Bmj} w^k{}_{,mj} + {}^{12}\mathcal{A}_Q^{Bkmj} e_k^Q{}_{,mj} = {}^1\mathcal{C}^B. \quad (74)$$

$$\text{dissipation inequality:} \quad {}^1\mathcal{B}_k^{mj} w^k{}_{,mj} + {}^2\mathcal{B}_Q^{kmj} e_k^Q{}_{,mj} \geq \mathcal{D}. \quad (75)$$

The coefficients ${}^{11}\mathcal{A}$, ${}^{12}\mathcal{A}$, ${}^1\mathcal{C}$, ${}^1\mathcal{B}$, ${}^2\mathcal{B}$ and \mathcal{D} can be directly read off from (71) and (73).

The spin balance (51) and the field equations (53) can be treated in the same manner as the energy-momentum balance and the dissipation inequality. Their symbolic shape is as follows

$$\text{spin:} \quad {}^{21}\mathcal{A}_k^{ABmj} w^k{}_{,mj} + {}^{22}\mathcal{A}_Q^{ABkmj} e_k^Q{}_{,mj} = {}^2\mathcal{C}^{AB}, \quad (76)$$

$$\text{field equations:} \quad {}^{31}\mathcal{A}_k^{ABmj} w^k{}_{,mj} + {}^{32}\mathcal{A}_Q^{ABkmj} e_k^Q{}_{,mj} = {}^3\mathcal{C}^{AB}. \quad (77)$$

The definitions of ${}^{21}\mathcal{A}$, ${}^{22}\mathcal{A}$, ${}^{31}\mathcal{A}$, ${}^{32}\mathcal{A}$, ${}^2\mathcal{C}^{AB}$ and ${}^3\mathcal{C}^{AB}$ can be obtained from (51) and (53) by the same treatment which was performed with the energy-momentum balance. We will not perform this procedure here, because our main aim is to demonstrate that an exploitation of the dissipation inequality is also possible in General Relativity Theory.

In the sequel we need the balances of energy-momentum and spin, the gravitational field equations and the dissipation inequality –(74) to (77)– in a common matrix formulation

$$\begin{pmatrix} {}^{11}\mathcal{A}_k^{Bmj} & {}^{12}\mathcal{A}_Q^{Bkmj} \\ {}^{21}\mathcal{A}_k^{ABmj} & {}^{22}\mathcal{A}_Q^{ABkmj} \\ {}^{31}\mathcal{A}_k^{ABmj} & {}^{32}\mathcal{A}_Q^{ABkmj} \end{pmatrix} \begin{pmatrix} w^k{}_{,mj} \\ e_k^Q{}_{,mj} \end{pmatrix} = \begin{pmatrix} {}^1\mathcal{C}^B \\ {}^2\mathcal{C}^{AB} \\ {}^3\mathcal{C}^{AB} \end{pmatrix} \quad (78)$$

$$\begin{pmatrix} {}^1\mathcal{B}_k^{mj} & {}^2\mathcal{B}_Q^{kmj} \end{pmatrix} \begin{pmatrix} w^k{}_{,mj} \\ e_k^Q{}_{,mj} \end{pmatrix} \geq \mathcal{D}. \quad (79)$$

The dissipation inequality (79) enforces that the \mathcal{A} , \mathcal{B} , \mathcal{C} and \mathcal{D} are not arbitrary. The exploitation of the dissipation inequality, the Liu procedure, is considered in the next section.

5 Exploitation of the Dissipation Inequality

The usual way to introduce the Second Law –the dissipation inequality (17)– into the exploitation of the balance equations –here (14), (29), (30) and (34)– is to solve this system of coupled differential equations and afterwards to check, if the dissipation inequality (17) is satisfied for all events x^a . Apart from the complexity of this procedure, it is not possible, because for solving the balances, we need constitutive equations determining the constitutive fields (56). But here, we are not interested in solving these balances, but we are looking for general restrictions to the constitutive equations by the dissipation inequality.

5.1 The Liu procedure

On account of the unknown constitutive equations, we cannot solve the balance equations, but we can ask for the conditions which must be satisfied by the constitutive equations, so that the dissipation inequality is locally valid for all events. The Liu procedure is such a tool for exploiting the dissipation inequality with regard to the balance and field equations resulting in restrictions for the constitutive equations [1, 2]²¹. The procedure removes the *higher derivatives* which are outside of the state space, and it generates constraints for the derivatives of the constitutive properties (56) in the state space. Here, we present the Liu procedure in a symbolical formalism.

The balances of energy-momentum (29) and of spin (30), the field equations (34) and the dissipation inequality (16) have the following shape

$$\boxtimes_{|C}^{AC}(\Delta^M) = \otimes^A, \quad \boxplus_{|C}^C(\Delta^M) \geq \oplus. \quad (80)$$

Here, Δ^M is the symbol for the state space variables (z, ζ) in (58).

Starting out with $(80)_1$, we obtain according to (68)

$$\boxtimes_{|C}^{AC} = \boxtimes_{,j}^{AC} e_C^j + \{ \boxtimes_C^{AC} \} = \otimes^A. \quad (81)$$

We now split the state space variables

$$\Delta^M = (\Delta_-^M, \Delta_+^M) \quad (82)$$

into those ones Δ_+^M whose derivatives $\Delta_{+,j}^M$, the higher derivatives, are out of the state space. Then, we split (81) into that part which is linear in the

²¹An other procedure to take the Second Law into consideration is that of Coleman and Noll [30].

higher derivatives, and into the remaining one which forms a part of the rhs of the following equation

$$e_C^j \frac{\partial \boxtimes^{AC}(\Delta^M)}{\partial \Delta_+^M} \Delta_{+,j}^M = -e_C^j \frac{\partial \boxtimes^{AC}(\Delta^M)}{\partial \Delta_-^M} \Delta_{-,j}^M - \{\boxtimes_C^{AC}\} + \otimes^A =: \boxdot^A. \quad (83)$$

Analogously, the dissipation inequality (80)₂ results in

$$e_C^j \frac{\partial \boxplus^C(\Delta^M)}{\partial \Delta_+^M} \Delta_{+,j}^M \geq -e_C^j \frac{\partial \boxplus^C(\Delta^M)}{\partial \Delta_-^M} \Delta_{-,j}^M - \{\boxplus_C^C\} + \oplus =: \odot. \quad (84)$$

Now we formulate

■ Liu's Theorem[1, 2]: There are functions of the state space variables $\Lambda_A(\Delta^M)$ generating the *Liu equations*

$$\Lambda_A e_C^j \frac{\partial \boxtimes^{AC}}{\partial \Delta_+^M} = e_C^j \frac{\partial \boxplus^C}{\partial \Delta_+^M} \longrightarrow \Lambda_A \frac{\partial \boxtimes^{AB}}{\partial \Delta_+^M} = \frac{\partial \boxplus^B}{\partial \Delta_+^M} \quad (85)$$

and the *reduced dissipation inequality*

$$\Lambda_A \boxdot^A \geq \odot, \quad (86)$$

removing the higher derivatives $\Delta_{+,j}^M$. ■

Liu equations and the reduced dissipation inequality represent constraints for the constitutive quantities \boxtimes^{AC} , \boxplus^C , \otimes^A and \oplus which we derive in the next section.

5.2 Liu equations and reduced dissipation inequality

According to (83) and (84), the lhs of the matrix equations (78) and (79) are linear in the higher derivatives. Consequently, we can apply the Liu procedure which according to (85)₂ and (86) results in the following matrix equations

$$\begin{pmatrix} {}^1\Lambda_B & {}^2\Lambda_{AB} & {}^3\Lambda_{AB} \end{pmatrix} \begin{pmatrix} {}^{11}\mathcal{A}_k^{Bmj} & {}^{12}\mathcal{A}_Q^{Bkmj} \\ {}^{21}\mathcal{A}_k^{ABmj} & {}^{22}\mathcal{A}_Q^{ABkmj} \\ {}^{31}\mathcal{A}_k^{ABmj} & {}^{32}\mathcal{A}_Q^{ABkmj} \end{pmatrix} = \begin{pmatrix} {}^1\mathcal{B}_k^{mj} & {}^2\mathcal{B}_Q^{kmj} \end{pmatrix} \quad (87)$$

$$\begin{pmatrix} {}^1\Lambda_B & {}^2\Lambda_{AB} & {}^3\Lambda_{AB} \end{pmatrix} \begin{pmatrix} {}^1\mathcal{C}^B \\ {}^2\mathcal{C}^{AB} \\ {}^3\mathcal{C}^{AB} \end{pmatrix} \geq \mathcal{D}. \quad (88)$$

Thus, we obtain the following Liu equations

$${}^1\mathcal{B}_k^{mj} = {}^1\Lambda_B {}^{11}\mathcal{A}_k^{Bmj} + {}^2\Lambda_{AB} {}^{21}\mathcal{A}_k^{ABmj} + {}^3\Lambda_{AB} {}^{31}\mathcal{A}_k^{ABmj}, \quad (89)$$

$${}^2\mathcal{B}_Q^{kmj} = {}^1\Lambda_B {}^{12}\mathcal{A}_Q^{Bkmj} + {}^2\Lambda_{AB} {}^{22}\mathcal{A}_Q^{ABkmj} + {}^3\Lambda_{AB} {}^{32}\mathcal{A}_Q^{ABkmj}, \quad (90)$$

and the reduced dissipation inequality

$${}^1\Lambda_B {}^1\mathcal{C}^B + {}^2\Lambda_{AB} {}^2\mathcal{C}^{AB} + {}^3\Lambda_{AB} {}^3\mathcal{C}^{AB} \geq \mathcal{D}. \quad (91)$$

According to (74) to (77), ${}^1\bullet\mathcal{A}$ belongs to the energy-momentum balance, ${}^2\bullet\mathcal{A}$ to the spin balance and ${}^3\bullet\mathcal{A}$ to the gravitational field equations. The same is valid for the $\bullet\mathcal{C}$. This division allows a classification into special cases which some of them are discussed in sect.6.

5.3 Constraints of state space variables

The particle number balance (14)

$$N_{|A}^A = 0 = (nu^A)_{|A} = n_{|A}u^A + nu_{|A}^A \quad (92)$$

and also (55) represent according to (64) a constraint for the state space variables. How to handle these constraints with respect to the Liu procedure, is expressed in the following

■ **Proposition**[31, 32, 33]: Constraints of the state space variables do not influence the Liu procedure, that means, the results achieved by the Liu procedure are independent of introducing the constraints before or after performing it. ■

Consequently, the particle number balance and the normalization of the 4-velocity do not generate additional constitutive restrictions.

6 Special Cases

6.1 Ignoring the field equations

The Mathisson-Papapetrou equations (29) and (30) are compatible with Einsteins field equations (34). Ignoring the field equations means that the influence of the material on the geometry is not taken into account: the curvature in (29) is given ad-hoc, but compatible with the field equations. Decoupling the field equation means, we have to ignore in (87) and (88) all quantities having the index combination ${}^3\bullet\odot$. The Liu equations (89) and (90) become

$${}^1\overset{\square}{\mathcal{B}}_k^{mj} = {}^1\overset{\square}{\Lambda}_B {}^{11}\mathcal{A}_k^{Bmj} + {}^2\overset{\square}{\Lambda}_{AB} {}^{21}\mathcal{A}_k^{ABmj}, \quad (93)$$

$${}^2\overset{\square}{\mathcal{B}}_Q^{kmj} = {}^1\overset{\square}{\Lambda}_B {}^{12}\mathcal{A}_Q^{Bkmj} + {}^2\overset{\square}{\Lambda}_{AB} {}^{22}\mathcal{A}_Q^{ABkmj}, \quad (94)$$

and the reduced dissipation inequality is

$${}^1\overset{\square}{\Lambda}_B {}^1\mathcal{C}^B + {}^2\overset{\square}{\Lambda}_{AB} {}^2\mathcal{C}^{AB} \geq \overset{\square}{\mathcal{D}}. \quad (95)$$

The constraints on the entropy –(93) and (94)– and on the entropy production (95) are changed with respect to the case, that the field equations are taken into consideration –(89) to (91). Ignoring the field equations results in altered material properties, if the Second Law is taken into account by the Liu procedure.

6.2 Vanishing spin

If the spin tensor is set to zero, we obtain from (27), (26) and (34)

$$T^{[AB]} = 0, \quad T_{|A}^{(AB)} = 0, \quad \frac{2}{\kappa} R^{AB} = 2T^{(AB)} - \eta^{AB} T_D^D. \quad (96)$$

The spin balance is satisfied by (96)₁ and (96)₂. Consequently, we ignore all quantities with the index combination $2^\bullet \odot$. The energy-momentum balance is (71) with the restriction that only the symmetric part ($AB \rightarrow (AB)$) comes into play and that S^{CDE} is set to zero. Thus, ${}^{11}\mathcal{A}$, ${}^{12}\mathcal{A}$ and ${}^1\mathcal{C}$ can be read off from (71).

The gravitational field equations (96)₃ become with (22)

$$\frac{2}{\kappa} G_{ADBQ}^{Dkmj} e_k^Q{}_{,mj} = -H_{ADB}^D + 2T_{(AB)} - \eta_{AB} T_D^D. \quad (97)$$

To take into consideration the Second Law, the following matrix elements vanish for state spaces of first order

$${}^{21}\mathcal{A}, {}^{22}\mathcal{A}, {}^2\mathcal{C}, {}^{31}\mathcal{A} \quad \text{are zero,} \quad (98)$$

and we obtain

$${}^{32}\mathcal{A}_{ABQ}^{kmj} = \frac{2}{\kappa} G_{ADBQ}^{Dkmj} \quad (99)$$

$${}^3\mathcal{C}_{AB} = -H_{ADB}^D + 2T_{(AB)} - \eta_{AB} T_D^D. \quad (100)$$

Consequently, the Liu equations (89) and (90) become

$${}^1\mathcal{B}_k^{mj} = {}^1\Lambda_B {}^{11}\mathcal{A}_k^{Bmj}, \quad (101)$$

$${}^2\mathcal{B}_Q^{kmj} = {}^1\Lambda_B {}^{12}\mathcal{A}_Q^{Bkmj} + {}^3\Lambda_{AB} {}^{32}\mathcal{A}_Q^{ABkmj}, \quad (102)$$

and the reduced dissipation inequality is

$${}^1\Lambda_B {}^1\mathcal{C}^B + {}^3\Lambda_{AB} {}^3\mathcal{C}^{AB} \geq \mathcal{D}. \quad (103)$$

This example of the Mathisson-Papapetrou equations demonstrates again that the dissipation inequality and its exploitation have to be supplemented, thus taking the Second Law into account. Even in the easy case of spinless material, the Second Law cannot be ignored independently of taking the gravitational field equations in the course of the Liu procedure into account or not.

6.3 Curvature insensitive material

A material is called *curvature insensitive*, if its material properties do not depend on $e_k^Q{}_{,m}$. In that case, the material does not contribute to the curvature,

because the last term vanishes in (70) and with it the influence of the material upon the curvature. Of course, the type of curvature sensitive material appear, always if the curvature comes into play.

The curvature does not explicitly appear in the spin balance (30) and in the dissipation inequality (17) or (52). Consequently, the matrix elements ${}^{22}\mathcal{A}$ and ${}^2\mathcal{B}$ vanish identically for curvature insensitive material according to (78) and (79). According to (90), we obtain

$$0 = {}^1\Lambda_B {}^{12}\mathcal{A}_Q^{Bkmj} + {}^3\Lambda_{AB} {}^{32}\mathcal{A}_Q^{ABkmj}, \quad (104)$$

an expression which also for curvature insensitive materials depends on the curvature because the curvature appears explicitly in the energy-momentum balance (29) and in the field equations (31).

The entropy production (91) may be influenced by the curvature: if the Λ depend on the curvature, the entropy production depends on it, too.

7 Discussion

The balance equations of continuum physics are formulated without taking a special material into consideration, that means, one needs constitutive equations to obtain by them a system of differential equations which is not under-determined. Constitutive equations are not arbitrary: they must be modeled in such a way, that the solution of the resulting system of differential equations satisfies the so-called material axioms. Among other axioms, the Second Law may be the most prominent one. In continuum physics, it runs as follows: Constitutive equations must have the property that the entropy production is never negative for all times and positions. A tool for finding such suitable constitutive equations is the Liu procedure which is here performed in the framework of General Relativity Theory.

The characteristic of general-relativistic continuum physics is, that beyond the balance equations, Einstein's gravitational field equations have to be taken into account. In accordance with Maugin's view described in footnote 5, we formally consider Einstein's gravitational equations on the same footing as the balance equations of continuum thermodynamics, rendering to apply the Liu procedure for the system of equations consisting of the energy-momentum and spin balances –(15) and (16)– together with Einstein's equations (18).

Compatibility of the balances with the field equations enforces, that the sources of the energy-momentum balance and of the spin balance are not arbitrary, but result in the Mathisson-Papapetrou equations, independently of including the field equations into the Liu procedure or not. As expected, the constraints for the constitutive equations are different, if the gravitational field equations are included into the Liu procedure or are ignored. Two other special cases are shortly discussed: the spin-less system in General Relativity and curvature

insensitive materials which are the only materials in Special Relativity, but a special case in GRT.

In the present paper, the Liu procedure is performed for material mappings defined on a first order state space. This results in the Liu conditions –(89) and (90)– that must be satisfied in order to guarantee the compatibility of the combined system of differential equations –(15), (16) and (18)– with the Second Law of thermodynamics (17).

Due to the complicated structure of the Liu equations in GRT, it is extremely difficult to arrive at general statements with regard to the compatibility of the general-relativistic continuum mechanics with the Second Law. Starting out with a constitutive equation by a substantiate guess, the Liu conditions –(89) to (91)– differentiate between gravitational fields as solutions of Einstein’s equations: there could exist solutions for which the constraints on the constitutive equations can be satisfied and such ones for which this cannot be achieved. Regarding the compatibility of the solutions of Einstein’s equations with continuum thermodynamics as an essential physical requirement, the Liu conditions do not qualify all solutions of the gravitational equations as to be physically relevant. In this sense, the Liu conditions act as a censor for the constitutive equations with respect to the Second Law.

Because the covariant curvature tensor (22) cannot be formulated covariantly as a function of the state space variables (see: sect.8), consequently the Liu procedure cannot be performed in a covariant manner. That is no disadvantage bearing into mind, that special solutions of the gravitational equations cannot generally be characterized covariantly, either.

8 Appendix

Curvature tensor in tetrad representation

Starting out with (3)₁, we obtain from (12) that the connexion with respect to the metric is a function of the tetrads and their first partial derivatives

$$\begin{aligned} \Gamma_{mi}^b(e_{a,b}^F, e_a^F) &= \\ &= \frac{1}{2}\eta_{AB}\eta^{DE}\left\{e_q^B(e_{m,i}^A + e_i^A{}_{,m}) + e_m^A(e_q^B{}_{,i} - e_i^B{}_{,q}) + e_i^A(e_q^B{}_{,m} - e_m^B{}_{,q})\right\}e_D^b e_E^q \end{aligned} \quad (105)$$

We obtain from (12) by partial differentiation

$$\Gamma_{mi,n}^b = \frac{1}{2}(g_{mq,in} + g_{iq,mn} - g_{mi,qn})g^{bq} + \Gamma_{mi}^p g_{pq} g^{bq}{}_{,n}, \quad (106)$$

resulting in

$$\begin{aligned} \Gamma_{mi,n}^b - \Gamma_{mn,i}^b &= \\ &= \frac{1}{2}(g_{iq,mn} - g_{mi,qn} - g_{nq,mi} + g_{mn,qi})g^{bq} + g_{pq}(\Gamma_{mi}^p g^{bq}{}_{,n} - \Gamma_{mn}^p g^{bq}{}_{,i}). \end{aligned} \quad (107)$$

Taking $(3)_1$ and

$$g_{mn,qi} = \eta_{AB}(e_{m,qi}^A e_n^B + e_{n,qi}^A e_m^B + e_{m,q}^A e_{n,i}^B + e_{m,i}^A e_{n,q}^B) \quad (108)$$

into account, we obtain that the curvature tensor depends linearly on the second derivatives of the tetrads. Consequently, it has the form which is given in (21). Some easy, but long-winded calculations result in

$$E_{mniG}^{buvw} = \eta_{AB}\eta^{CD}e_C^b e_D^q \left[(\delta_i^w e_n^B - \delta_n^w e_i^B)(\delta_G^A \delta_m^u \delta_q^v - \delta_G^A \delta_q^u \delta_m^v) + \right. \\ \left. + (\delta_n^v e_q^A - \delta_q^v e_n^A)(\delta_G^B \delta_i^u \delta_n^w - \delta_G^B \delta_n^u \delta_i^w) \right], \quad (109)$$

$$F_{mni}^b = \Gamma_{mi}^j \Gamma_{nj}^b - \Gamma_{mn}^j \Gamma_{ij}^b + g_{pq}(\Gamma_{mi}^p g^{bq},_n - \Gamma_{mn}^p g^{bq},_i) + \\ + \eta_{AB} g^{bq} \left[(e_{m,q}^A - e_{q,m}^A)(e_i^B,{}_n - e_n^B,{}_i) + \right. \\ \left. + e_{m,n}^A e_i^B,{}_q - e_{m,i}^A e_n^B,{}_q + e_{q,i}^A e_n^B,{}_m - e_{q,n}^A e_i^B,{}_m \right]. \quad (110)$$

The final result is generated by inserting $(3)_1$ and (105) into (110).

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